Canonical Phase Space Formulation of Quasi-local General Relativity

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Abstract. We construct a Hamiltonian formulation of quasi-local general relativity using an extended phase space that includes boundary coordinates as configuration variables. This allows us to use Hamiltonian methods to derive an expression for the energy of a non-isolated region of space-time that interacts with its neighbourhood. This expression is found to be very similar to the Brown-York quasi-local energy that was originally derived by Hamilton-Jacobi methods. We examine the connection between the two formalisms and find that when the boundary conditions for the two are harmonized, the resulting quasi-local energies are identical.

1. Introduction

General relativity is a fully covariant theory of gravity and as such does not privilege any particular flow of time. This is one of its great virtues but it is also a problem if one wishes to study it using traditional methods that manifestly depend on a notion of time. For example, a flow of time must be defined before one can do a standard Hamiltonian phase space analysis of a field theory. Thus, if we want to apply such methods to Einstein's gravity, space-time must be artificially broken into space and time. We do this by foliating space-time into a set of space-like three-surfaces ("instants of time") Σ_t, and defining a "time-flow" vector field T^a that evolves these surfaces into each other.

Having done this, one can reformulate general relativity in terms of phase space, symplectic structures, and Hamiltonians. In the most common approach, the space-like three-metric h_{ab} is taken as the configuration variable while its conjugate momentum, P^{ab} is closely related to the extrinsic curvature of the three-surface in four-dimensional space-time (see for example [1, 2, 3]). Traditionally, space-times studied in this way were either boundary-free or taken to have a boundary at spatial or null infinity (with appropriate conditions imposed so that the space-time is asymptotically flat). More recently, people have become interested in studying general relativity over finite regions of space-time, which are often referred to as quasi-local regions. Then, boundaries and boundary conditions at locations other than infinity must be considered. In [4, 5], the assorted boundary conditions that give rise to a phase space with a well-defined symplectic structure were studied. In particular, it is well known that this can be done if one fixes the intrinsic metric γ_{ab} of the boundary to be constant and not affected by variations. That is, one fixes the intrinsic geometry at each point of the boundary manifold.

However, some aspects of this fix are not entirely satisfactory. For example, if the fixed boundary metric is not axially symmetric, then a rotation along a vector field ϕ which is tangent to the boundary at the boundary *is not* an allowed variation in the phase space under consideration, because $\delta_\phi \gamma_{ab} = \mathcal{L}_\phi \gamma_{ab} \neq 0$ and so the variation does not preserve the boundary condition. Thus, there is no Hamiltonian generating this motion in phase space and consequently no notion of the angular momentum associated to such a rotation. Similarly, time translations are not permitted unless the boundary metric is invariant in time, i.e. $\delta_T \gamma_{ab} = \mathcal{L}_T \gamma_{ab} = 0$. These evolutions cannot be generated by Hamiltonian functionals unless ϕ^a and T^a are Killing vector fields of γ_{ab}.

A little thought shows that this is really not so surprising. In both of the examples discussed above, the "conserved" quantity corresponding to the listed translation is not conserved. Hence, we should not expect to be able to obtain it from a standard Hamiltonian treatment, which by its very nature applies to situations where Hamiltonians are conserved. For example, if a Hamiltonian exists that generates time translations, then Hamilton's

equations are

$$\delta H_T = \Omega(\delta_T, \delta) \tag{1}$$

where the symplectic structure Ω is antisymmetric in the two variations. Hence, it follows immediately that $\delta_T H_T = 0$. Similarly, the Hamiltonian generating a particular rotation is conserved under that same rotation.

Despite the arguments given above, Brown and York have introduced notions of the energy and angular momentum associated with a bounded region of space-time (if the intrinsic boundary metric is fixed) by performing a Hamilton–Jacobi analysis [6]. Furthermore, the expressions they have obtained have proved useful in a very large number of applications (as a representative sample see [8]). It would be very surprising if these expressions could not be derived from a phase space treatment of general relativity over a manifold with boundary.

The purpose of this paper is to show that it is indeed possible to obtain these expressions for energy and angular momentum using a careful phase space analysis of general relativity on manifolds with boundaries. Here, we begin with the standard phase space formulation of general relativity in terms of ADM variables (see, for example, [9] for details). The key idea is then to import extended phase space techniques from classical mechanics which are designed to deal with situations where "conserved" quantities are not conserved. The basic idea is to enlarge the phase space under consideration by including quantities such as the time coordinate as configuration variables (see [10] for a standard reference). It is then possible to construct a conserved Hamiltonian generating time translations in the extended phase space. However, the energy of the system is no longer the value of this Hamiltonian, but is instead the value of the momentum which is canonically conjugate to time.

In addition to extending the phase space however, it is also necessary to weaken the boundary conditions so that time translations and rotations *are* allowed variations. For example, in the Brown-York treatment the boundary three-metric γ_{ab} is completely fixed on a three-manifold B that is a boundary of the space-time manifold M. Therefore,

$$\delta\gamma_{ab} = 0 \tag{2}$$

for all variations. That is, the metric is fixed with respect to the manifold. Under our looser treatment, we allow the variations to act as diffeomorphisms that move our boundary fields around B. Thus, δ acts on the "fixed" fields as an infinitesimal diffeomorphism and so

$$\delta\gamma_{ab} = \mathcal{L}_{\delta X}\gamma_{ab}, \tag{3}$$

for some vector field δX^a. Picking δX^a appropriately, we obtain translations and rotations as variations.

The generalization of the Hamiltonian formulation for a finite region of space-time is, to our knowledge, presented here for the first time. However, similar ideas have previously

been discussed at spatial infinity. Notably, in their early work on boundary terms for the gravitational Hamiltonian, Regge and Teitelboim [2] implemented a similar programme for the boundary at spatial infinity in asymptotically flat space-times. Specifically they included variables that located the asymptotic position of the boundary as well as their conjugate momenta as canonical variables and showed that the resulting Hamiltonian was covariant with respect to the asymptotic Poincaré group. Kuchař has also studied parameterizations at infinity. In his considerations of Hamiltonian formulations for spherically symmetric black holes [7], he included the Killing time as a canonical coordinate and found that the mass was conjugate to its radial rate of change. Finally, Kijowski [5] has also considered general relativity in a manifold with boundary. He begins with a novel and non-standard approach to symplectic geometry and the Hamiltonian formulation [11]. Despite this different starting point, he obtains energy expressions for the boundary which are analogous to the Brown–York. However, he only explicitly considers the case where the evolution is a symmetry of the boundary. Thus, our work may be considered a generalization to arbitrary boundary geometries. It is likely that our method could also be incorporated into this alternative formalism.

The algebra for the quasi-local gravitational case is quite formidable, and so as an introduction, we begin in section 2 with a simple example of the extended phase space formalism, namely a time dependent harmonic oscillator. With this experience in hand, in section 3 we apply similar techniques to the problem of interest: general relativity in a space-time manifold with a boundary. In particular, we obtain expressions for the energy and angular momentum associated to the region of space-time, which may be calculated using just the values of the fields at that boundary. Section 4 provides a comparison between our results and those of Brown and York. We end with a discussion of the results and possible extensions and applications of this work. Several key technical results are collected in the appendices.

2. An Introductory Example

In this section, we shall consider a simple example which will demonstrate how a phase space can be extended to allow the description of systems in which the energy is not constant. By starting with a time-dependent harmonic oscillator we will capture many of the central ideas of the construction without the extra complications that arise in gravity. The key idea will be to extend the phase space by including the time coordinate t as a configuration variable as well as its conjugate momentum p_t. Then if we know how the variables evolve in time, we can manipulate the symplectic structure so as to find a Hamiltonian function which generates that evolution on-shell. Equivalently, we solve the Hamiltonian equations of motion for the given evolution to find the corresponding Hamiltonian.

With the standard phase space treatment, the energy of the system is the on-shell value of this generator of time translations (which is, of course, constant). In the extended phase space treatment, the energy associated with a time translation is equal to the negative of the value of the momentum conjugate to t. A slight complication is that, in general, the Hamiltonian is not unique and as a result of this ambiguity the energy conjugate to the evolution is not unique either. Indeed a significant degree of freedom remains in its definition. We illustrate and elaborate on these issues in the following example.

2.1. Simple harmonic oscillator

Consider the harmonic oscillator with mass m and spring constant k. The canonical phase space of this system is parameterized by the position coordinate q and its conjugate momentum p. The symplectic structure is simply given by

$$\Omega(\delta_1, \delta_2) = (\delta_1 q)(\delta_2 p) - (\delta_2 q)(\delta_1 p) \,. \tag{4}$$

In this and all future expressions, one should keep in mind that the δs are vectors in the phase space of all possible configurations of the system. Their conventional interpretation as infinitesimal variations of system configurations arises from considering the one-parameter families of phase space diffeomorphisms that are generated by such vector fields. "Infinitesimal" changes of those parameters generate what we intuitively think of as infinitesimal variations of the system.

Then, as discussed in the introduction, a phase space vector field δ_t (equivalently an evolution of the system) is said to be Hamiltonian if there exists a function H_t such that

$$\Omega(\delta_t, \delta) = \delta H_t \,, \tag{5}$$

for all variations δ. Conversely, given a Hamiltonian function H_t, we can find out how it evolves the system by solving the above equation for δ_t. Thus, there is a mapping between Hamiltonian evolutions and Hamiltonian functions.

As an example, for the evolution δ_t which gives rise to the usual equations of motion:

$$\frac{dq}{dt} = \frac{p}{m} \quad \text{and} \quad \frac{dp}{dt} = -kq, \tag{6}$$

we can show that

$$\Omega(\delta_t, \delta) = \left(\frac{p}{m}\right) \delta p - (-kq)\delta q = \delta H_t, \tag{7}$$

where

$$H_t = \frac{p^2}{2m} + \frac{kq^2}{2} + C \,, \tag{8}$$

and C is a free constant. Thus, the time evolution is generated by H_t, which the reader will immediately identify as the classical energy of the system (up to a constant). Note too that the equations of motion confirm that $\delta_t H_t = 0$ (which we knew already by the skew-symmetry of the symplectic structure).

2.2. Time-dependent simple harmonic oscillator

Things get more complicated if k is not a constant, but instead varies in time. In this case, the energy of the oscillator will not be constant, but instead may vary as a changing k adds energy to or removes it from the system (more properly, the external agency setting k can do net work on the system). Thus, if we have no knowledge of how k is being fixed, we cannot do a standard Hamiltonian analysis of the system – such calculations require a closed system which does not exchange energy with its surroundings. However, we can use the extended phase space formalism to partially compensate for our ignorance of the external system that sets k. All we require is that $k(t)$ be a fixed function of t.

To allow for a time dependent spring constant k and consequently a time dependent energy, it is necessary to extend the phase space to include t and its conjugate momentum p_t. The symplectic structure is then given by

$$\Omega(\delta_1, \delta_2) = (\delta_1 q)(\delta_2 p) - (\delta_2 q)(\delta_1 p) + (\delta_1 t)(\delta_2 p_t) - (\delta_2 t)(\delta_1 p_t). \tag{9}$$

Furthermore, we would like to allow general evolutions, rather than restricting to (d/dt). Thus, we shall study evolution generated by

$$\Lambda = \lambda_o \frac{d}{dt}\,, \tag{10}$$

where λ_o is a free parameter which is required to be strictly positive. Our task is then to find a Hamiltonian ‡K_Λ which generates the following evolution:

$$\delta_\Lambda q = \lambda_o \frac{p}{m}, \quad \delta_\Lambda p = -\lambda_o k q, \quad \text{and} \quad \delta_\Lambda t = \lambda_o. \tag{11}$$

Note that the evolution of p_t under δ_Λ is not determined a priori. This in turn leads to a freedom in the form of the Hamiltonian K_Λ.

We proceed by evaluating $\Omega(\delta_\Lambda, \delta)$. Making use of (11), as well as the fact that $k(t)$ is a fixed function of t so that

$$\delta k = \dot{k}\, \delta t$$

we obtain

$$\begin{aligned} \Omega(\delta_\Lambda, \delta) &= \lambda_o\, \delta \left(p_t + \frac{p^2}{2m} + \frac{kq^2}{2} \right) - \left(\frac{\lambda_o q^2 \dot{k}}{2} + \delta_\Lambda p_t \right) \delta t \\ &= \delta(\lambda_o K_t) - K_t\, \delta\lambda_o - \left(\frac{\lambda_o q^2 \dot{k}}{2} + \delta_\Lambda p_t \right) \delta t\,, \end{aligned} \tag{12}$$

where we have defined

$$K_t := p_t + \frac{p^2}{2m} + \frac{kq^2}{2}\,. \tag{13}$$

‡ We will follow the convention of [10] and use K to denote the Hamiltonian in extended phase space.

Now, our evolution δ_Λ will be a Hamiltonian vector field in the phase space if and only if $\Omega(\delta_\Lambda, \delta)$ is an exact variation. This will only be true if the $\delta\lambda_o$ and δt terms vanish. Therefore, we must be at a point in phase space where

$$K_t = 0 \quad \Rightarrow \quad p_t = -\left(\frac{p^2}{2m} + \frac{kq^2}{2}\right). \tag{14}$$

Furthermore, p_t must satisfy the equation of motion:

$$\delta_\Lambda p_t = -\frac{\lambda_o q^2 \dot{k}}{2}, \tag{15}$$

which guarantees that the constraint (14) is preserved under Λ-evolution. Then, on this constraint surface, the evolution is Hamiltonian and generated by

$$K_\Lambda = \lambda_o K_t. \tag{16}$$

The reader will immediately realize that this function vanishes on-shell. However, this does not mean that the energy of the system will vanish. In the extended phase space, the energy is given by the negative of the value of the momentum canonically conjugate to the time. Thus we obtain

$$E_t := -p_t = \frac{p^2}{2m} + \frac{kq^2}{2}. \tag{17}$$

This is immediately recognized as the usual energy associated to a harmonic oscillator, although it will not necessarily be constant due to the time dependence of $k(t)$.

The Hamiltonian K_Λ given in (16) generates the desired evolutions (11) of t, p, and q, but it is by no means the unique Hamiltonian that does this. With no $\delta_\Lambda p_t$ specified *a priori*, the evolution of p_t can take any form that we like, and this freedom translates into an ambiguity in both the Hamiltonian K_Λ and the energy E_t. We explore the range of possible Hamiltonians (and therefore energies) by considering the functions that may be added to K_Λ without affecting the evolution equations (11).

To start one would consider functions of all possible variables and parameters — that is functions of the form $f(t, p_t, q, p, \lambda_o)$. However, we immediately see that a dependence on p_t, q, or p will change the evolution equations (11). Therefore only functions of the form $f(t, \lambda_o)$ may be considered. For such functions, the constraint equation $K_t = 0$ transforms to become

$$p_t + E_t + \frac{\partial f}{\partial \lambda_o} = 0. \tag{18}$$

Next, we demand that the constraint equations do not depend on the Lagrange multiplier λ_o. This is equivalent to requiring that either the energy E_t should not depend on λ_o or equally the equations of motion should not restrict the allowed values (or evolution) of λ_o.

With this assumption, the freedom is reduced to $f(\lambda_o, t) = \lambda_o g(t) + h(t)$, for any functions $g(t)$ and $h(t)$. Then, the derived equation of motion for p_t (15) becomes

$$\delta_\Lambda p_t = -\lambda_o \frac{\dot{k}q}{2} - \frac{\partial f}{\partial t} \,. \tag{19}$$

This will only be consistent with (18) if $h(t)$ is in fact a constant C. Then any Hamiltonian of the form

$$K'_\Lambda = \lambda_o \left(p_t + \frac{p^2}{2m} + \frac{kq^2}{2} + g(t) \right) + C \,, \tag{20}$$

will be consistent with our requirements, and so the energy will only be defined up to a free function:

$$E'_t = \frac{p^2}{2m} + \frac{kq^2}{2} + g(t) \,. \tag{21}$$

We finish this analysis of the freedom using a physical argument. Mathematically, any function $g(t)$ will satisfy our requirements. Physically however, it is reasonable to demand that this function should be in some way connected to the system. With that requirement we are reduced to considering functions of $k(t)$.

3. Gravity

We now turn to a canonical Hamiltonian formulation of general relativity over a quasi-local region of space-time. Although the technical details will, of course, be much more complicated, many of the basic conceptual issues relating to extended phase space have already been dealt with in the previous example. Thus, for gravity we will also extend the usual phase space to include a time variable — which will be defined only on the boundary — and it will be joined by the spatial coordinates of the boundary. Further, just as k in the above was only fixed up to changes in the time parameter, for quasi-local gravity our boundary conditions will only be fixed up to intrinsic diffeomorphisms of that boundary. We will also find that a range of Hamiltonians will generate the standard evolutions of a space-time and that this freedom may be traced to the freedom to choose the evolution of the conjugate momenta of the boundary coordinates. Each of these Hamiltonians will be valid on its own constraint surface, and again each of these constraint surfaces will correspond to a different energy function for the system.

3.1. Manifold without boundary

Let us begin by briefly reviewing the Hamiltonian formulation of general relativity for a manifold with no boundaries. This will also serve to fix our notation and conventions. We will consider time-dependent fields living on a space-like three-manifold Σ. The intrinsic

geometry of Σ is fully specified by a space-like three-metric h_{ab}; the derivative operator compatible with the metric will be denoted by D_a. In the standard Hamiltonian formulation, the 3-metric h_{ab} serves as the configuration variable, while its conjugate momentum is the tensor density P^{ab}. Thus, the phase space consists of pairs of fields (h_{ab}, P^{ab}) with the symplectic structure

$$\Omega(\delta_1, \delta_2) = \int_\Sigma d^3x \left\{ (\delta_1 h_{ab})(\delta_2 P^{ab}) - (\delta_2 h_{ab})(\delta_1 P^{ab}) \right\} . \tag{22}$$

To specify a time evolution on Σ, we introduce a lapse function N and a shift vector field $V^a \in T\Sigma$. In the usual way the lapse and the shift will prescribe how time "flows" on Σ. Only after these fields are given can we define a time derivative $\frac{d}{dt}$ over Σ, as only then will we know how to associate points at a time t with points at time $t + \delta t$, and also know how much proper time has passed during that interval. Given a flow of time (or equivalently a lapse and shift), we introduce a Hamiltonian

$$H_t = \int_\Sigma d^3x \, \{ N\mathcal{H} + V^a \mathcal{H}_a \} \tag{23}$$

which generates time evolution. Specifically, we obtain

$$\frac{d}{dt} h_{ab} = [h_{ab}]_{(N,V)} \text{ and} \tag{24}$$

$$\frac{d}{dt} P^{ab} = [P^{ab}]_{(N,V)} , \tag{25}$$

where the exact forms of $[h_{ab}]_{(N,V)}$ and $[P^{ab}]_{(N,V)}$ are given (along with expressions for the Hamiltonian constraint $\mathcal{H}$ and diffeomorphism constraint $\mathcal{H}_a$) in Appendix A. Furthermore, the initial data for h_{ab} and P^{ab} must satisfy the constraints

$$\mathcal{H} = 0 \quad \text{and} \quad \mathcal{H}_a = 0 . \tag{26}$$

which are then automatically preserved in time.

3.2. Boundary Conditions

We would like to extend the Hamiltonian formulation of general relativity to manifolds with boundary. Therefore, we will now consider Σ to be a space-like 3-manifold with a closed 2-boundary $\mathcal{B}$. In this section, we will describe the boundary conditions enforced on $\mathcal{B}$. The essential idea is to keep the boundary metric, lapse and shift fixed, up to diffeomorphisms of the boundary. To make this precise, we proceed as follows.

Construct a three-manifold $B \cong \mathcal{B} \times \mathbb{R}$ which is foliated by two-manifolds $\mathcal{B}_{\bar{t}} \cong \mathcal{B}$ where $\bar{t}$ is the foliation parameter and "$\cong$" indicates that the manifolds are diffeomorphic. The parameter $\bar{t}$ provides a notion of time on B. Specifically, the $\mathcal{B}_{\bar{t}}$ are taken as "instants" of time, and $\bar{t}_2$ is said to occur after $\bar{t}_1$, if $\bar{t}_2 > \bar{t}_1$. We then introduce a time-like three-metric $\bar{\gamma}_{ab}$ on B which pulls back to a space-like two-metric $\bar{\sigma}_{ab}$ on each of the elements of the foliation.

Using the three-metric, we can also obtain the future-directed unit normal vector field to the $\mathcal{B}_{\bar{t}}$, which we denote $\bar{u}^a$. Finally, we introduce a time evolution vector field $\bar{T}^a \in TB$ which satisfies

$$\mathcal{L}_{\bar{T}} \bar{t} = 1 . \tag{27}$$

We note that $\bar{T}^a$ may be decomposed into its parts perpendicular and parallel to the $\mathcal{B}_{\bar{t}}$ as

$$\bar{T}^a = \bar{N}\bar{u}^a + \bar{V}^a , \tag{28}$$

where $\bar{N}$ is the lapse function and $\bar{V}^a \in T\mathcal{B}_{\bar{t}}$ is the shift vector on B. Then, the future pointing unit normal to $\mathcal{B}_{\bar{t}}$ is

$$\bar{u}_a = -\bar{N} d\bar{t}_a . \tag{29}$$

We can also decompose the metric $\bar{\gamma}_{ab}$ as

$$\bar{\gamma}_{ab} = -\bar{N}^2 d\bar{t}_a d\bar{t}_b + \bar{\sigma}_{ab} , \tag{30}$$

while

$$\bar{\gamma}^{ab} = -\frac{1}{\bar{N}^2}\bar{T}^a\bar{T}^b + \frac{1}{\bar{N}}\bar{T}^{(a}\bar{V}^{b)} + \bar{\sigma}^{ab}. \tag{31}$$

With this framework in place, we are ready to introduce our boundary condition on $\mathcal{B}$.

The Boundary Conditions

(i) Construct the 3-manifold B and equip it with a foliation, fixed time-like boundary metric $\bar{\gamma}_{ab}$ and time evolution vector field $\bar{T}^a$ as described above.

(ii) Introduce a smooth diffeomorphism

$$\omega : \mathcal{B} \times \mathbb{R} \to B, \tag{32}$$

so that if t is the time parameter for Σ and $\mathcal{B}$, and $\bar{t}(t)$ is some monotonically increasing function from $\mathbb{R} \to \mathbb{R}$, then

$$\omega(\cdot, t) : \mathcal{B} \to \mathcal{B}_{\bar{t}(t)}, \tag{33}$$

is a diffeomorphism for all $t \in \mathbb{R}$ (see figure 1). That is, instants of time on $\mathcal{B}$ are mapped to our pre-defined "instants" of time in B.

(iii) The two-metric σ_{ab}, lapse N, and shift V^a on $\mathcal{B}$ at a time t are equal to the corresponding $\bar{\sigma}_{ab}$, $\bar{N}$, and $\bar{V}^a$ pulled-back to $\mathcal{B}$ using $\omega(\cdot, t)$ (or pushed-forward using $\omega^{-1}(\cdot, \bar{t})$ in the case of $\bar{V}^a$).

(iv) The extrinsic curvature induced on B by h_{ab} and P^{ab} must satisfy the time-like diffeomorphism constraint.

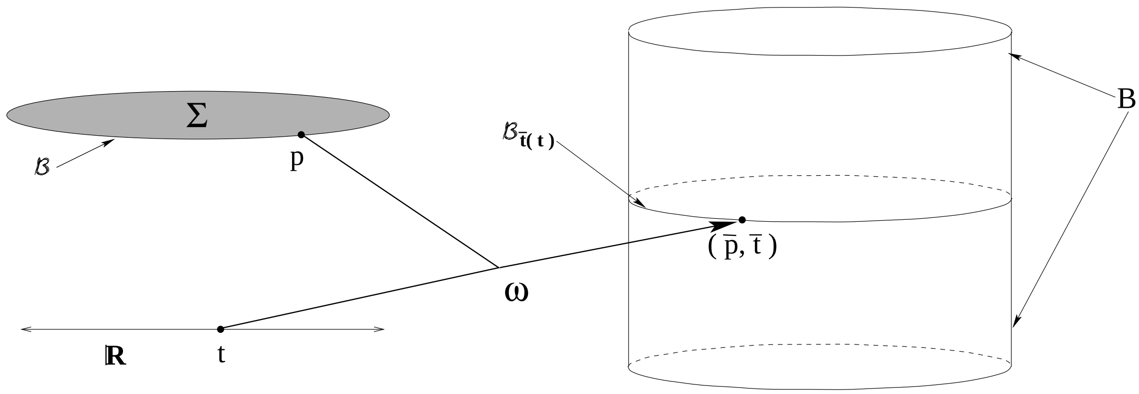

Figure 1. The map ω (with one dimension of $\mathcal{B}$ suppressed).

In condition (iii), we have required $V^a = \bar{V}^a$ which ensures $V^a \in T\mathcal{B}$ on $\mathcal{B}$. However, in the initial set-up, we only required that $V^a \in T\Sigma$. This extra restriction is equivalent to the orthogonality assumption of Brown-York (which is discussed in more detail in section 4) and is not essential but will somewhat simplify the already involved discussion that will follow. The more general non-orthogonal case, which only requires that $N^2 - V^a V_a = \bar{N}^2 - \bar{V}^a \bar{V}_a$ and is essential if we wish to allow B to be null or space-like, has been studied and will be discussed in a future paper. For now, however, we will work with this orthogonality assumption and so find that

$$\frac{d}{dt} = \mathcal{L}_{\bar{T}}\,, \tag{34}$$

when these operators act on fields that are defined on B and then mapped back to $\mathcal{B}$. Finally, (iv) says that even when we later consider general variations of our fields, they must continue to satisfy the diffeomorphism constraint at $\mathcal{B}$.

In order to describe the phase space of quasi-local general relativity, we need a concrete realization of the diffeomorphism ω. To obtain this, we provide a coordinatization of the three-manifold B in terms of the foliation parameter $\bar{t}$ and two "angular" coordinates $\bar{x}^A$ on $\mathcal{B}_t$ §, which are chosen to be "time-independent" — that is

$$\mathcal{L}_{\bar{T}} \bar{x}^A = 0. \tag{35}$$

(Here, and in the remainder of the paper, we will use capital letters to signify the coordinates and components of tensors in this chosen coordinate system, while lower case letters are the usual abstract index notation.) Once the coordinate system on B is given, ω may be described

§ In many cases, multiple coordinate systems will be required to cover B without coordinate singularities and corresponding singularities in the coordinate vector fields. The calculations of this paper extend directly to cover these situations, but in the interests of clarity, we'll proceed as if one set of angular coordinates was sufficient.

by how it pulls back the coordinate system to $\mathcal{B} \times \mathrm{I\!R}$. Giving $(\bar{x}^A, \bar{t})$ at each point in $\mathcal{B} \times \mathrm{I\!R}$ uniquely determines the map ω. Thus, we can (and do) specify ω by assigning an $\bar{x}^A$ and $\bar{t}$ to each time and place in $\mathcal{B}$.

Notice that while (ii) specifies that a diffeomorphism exists, the exact form of ω is not fixed. However, any permissible ω can be obtained from any other ‖ simply by composing ω with a diffeomorphism

$$\phi : B \to B \quad \text{mapping} \quad (\bar{p}, \bar{t}) \mapsto (\bar{p}', \bar{t}') = \phi(\bar{p}, \bar{t}) \,, \tag{36}$$

which preserves the foliation of B. Thus, the freedom in defining ω is the freedom to consider a preferred ω_o composed with all ϕ, where

$$\phi \circ \omega_o : \mathcal{B} \times \mathrm{I\!R} \to B \quad \text{maps} \quad (p, t) \mapsto \phi\left(\omega_o(p, t)\right) \,. \tag{37}$$

Since ω_o mapped all fields ($\bar{\sigma}_{ab}$, $\bar{N}$, $\bar{V}^a$, and the coordinates $\bar{t}$ and $\bar{x}^A$) back to $\mathcal{B}$, then $\phi \circ \omega_o$ will too. That is, from a computational point of view, our diffeomorphisms act so that observers fixed to points either on B or $\mathcal{B}$ will see all of these fields perturbed.

For an infinitesimal diffeomorphism ϕ, it is straightforward to calculate the changes to the various fields at the boundary. We begin by recalling that infinitesimal diffeomorphisms are generated by non-singular vector fields and so to each ϕ we may associate a vector field $\delta X = (\delta \bar{t}, \delta \bar{x}^a)$. The requirement that ϕ preserve the foliation means that $\delta \bar{t}$ should be a constant on each leaf of the foliation, although it can vary from one leaf to the next. Meanwhile, $\delta \bar{x}^a$ can be any non-singular vector field in TB that is everywhere parallel to the foliation surfaces. Under the infinitesimal action of this diffeomorphism the fields on the boundary B will change according to

$$\begin{aligned}
\bar{\gamma}_{ab} &\mapsto \bar{\gamma}_{ab} + \mathcal{L}_{\delta X} \bar{\gamma}_{ab} \\
d\bar{t} &\mapsto d\bar{t} + \mathcal{L}_{\delta X} d\bar{t} \\
\bar{T} &\mapsto \bar{T} + \mathcal{L}_{\delta X} \bar{T} \\
\bar{t} &\mapsto \bar{t} + \delta \bar{t} \quad \text{and} \\
\bar{x}^A &\mapsto \bar{x}^A + \delta \bar{x}^A .
\end{aligned}$$

Here, $\delta \bar{x}^a$ is simply the vector field $\delta \bar{x}$, expressed using the abstract index notation, while the $\delta \bar{x}^A$ are the *components* of the vector field $\delta \bar{x}$. It will be important to keep this distinction clear. The two are related by

$$\delta \bar{x}^A = (\delta \bar{x})^a d_a \bar{x}^A \,, \tag{38}$$

where d_a is the intrinsic derivative over $\mathcal{B}$.

‖ Up to potential global topological obstructions which we shall ignore with impunity since in our calculations we will only consider "infinitesimal" variations of ω.

The corresponding change to the fields in $\mathcal{B} \times \mathrm{I\!R}$ is given by

$$N \mapsto N + (\delta t)\frac{d}{dt}N + \mathcal{L}_{\delta x}N \tag{39}$$

$$V^a \mapsto V^a + (\delta t)\frac{d}{dt}V^a + \mathcal{L}_{\delta x}V^a \tag{40}$$

$$\sigma_{ab} \mapsto \sigma_{ab} + (\delta t)\frac{d}{dt}\sigma_{ab} + \mathcal{L}_{\delta x}\sigma_{ab}\,, \tag{41}$$

$$t \mapsto t + \delta t \quad \text{and} \tag{42}$$

$$x^A \mapsto x^A + \delta x^A, \tag{43}$$

after ω maps everything back to $\mathcal{B}$. Note that we have dropped the over-bars in order to reduce notational clutter. In the future we will blur the distinction between $\mathcal{B} \times \mathrm{I\!R}$ and B. Whenever there is an ambiguity, ω is understood to be acting to make the identification and map quantities back and forth.

To summarize, our boundary conditions imply that once a map ω from $\mathcal{B} \times \mathrm{I\!R}$ to B is specified, the boundary metric, lapse and shift are known. We have simplified matters by introducing a coordinate system on B which allows us to easily characterize the map ω, but the results obtained in the following subsections will not be sensitive to this coordinate system, and it is likely that the calculations could be done without introducing coordinates at all. What will be important is that the *only* allowed variations of the boundary will be infinitesimal diffeomorphisms. They will generate changes in the boundary metric, lapse and shift as given by (39-41). Additionally, they will change the map ω, or equivalently the coordinates associated to the points of $\mathcal{B}$, according to (42,43).

Comparing with earlier work, Regge and Teitelboim [2] showed that their Hamiltonian was covariant with respect to the Poincaré group at infinity acting on the boundary. Here we have set things up so that we may study the effect of the diffeomorphism group that maps the boundary into itself (while preserving the foliation) on the Hamiltonian. Thus, apart from the difference between boundaries at finite difference and infinity, we are also studying different group actions on those boundaries.

3.3. Phase Space and Hamiltonian Evolution

In the previous subsections, we considered the fields in the bulk as well as the boundary conditions imposed at $\mathcal{B}$. In this subsection, we will turn our attention to the phase space and associated symplectic structure, as well as the Hamiltonians associated with evolution equations.

With the boundary conditions that we have imposed, the configuration variables of a point in the phase space will be given by the three-metric h_{ab}, a value of the time parameter t, and a set of angular coordinates x^A on $\mathcal{B}$. The conjugate momenta to these configuration variables will be P^{ab}, P_t, and P_A, and so the coordinates of a point in phase space will be

given by the six fields $(h_{ab}, P^{ab}, t, P_t, x^A, P_A)$. Note that our boundary conditions say that once we specify t and the x^A, we will know σ_{ab} over $\mathcal{B}$, thus h_{ab} cannot be chosen completely independently of those variables. The symplectic structure ¶ on the phase space then takes the form:

$$\begin{aligned}\Omega(\delta_1, \delta_2) = & \int_\Sigma d^3x \left\{(\delta_1 h_{ab})(\delta_2 P^{ab}) - (\delta_2 h_{ab})(\delta_1 P^{ab})\right\} \\ & + \int_{\mathcal{B}} d^2x \left\{(\delta_1 x^A)(\delta_2 P_A) - (\delta_2 x^A)(\delta_1 P_A)\right\} \\ & + \{(\delta_1 t)(\delta_2 P_t) - (\delta_2 t)(\delta_1 P_t)\} \ .\end{aligned} \tag{44}$$

The variations which appear in the bulk are entirely free, and are not restricted to being on-shell. However, they are partially restricted at the boundary. Specifically, variations must preserve the boundary conditions, and so must be generated by some $(\delta t, \delta x^A)$. Then the variations of the lapse N, shift V^a, two-metric σ_{ab}, the time parameter t, and the coordinates x^A are given by equations (39-43).

A typical on-shell variation δ_Λ will be generated by the action of

$$\Lambda = \lambda_o \frac{d}{dt} + \mathcal{L}_\lambda, \tag{45}$$

where λ_o must be constant on the boundary (so as to preserve the foliation of the boundary) and $\lambda \in T\Sigma_t$ in the bulk while it is tangent to $\mathcal{B}_t$ on the boundary. Then we know how our phase space variables must evolve with Λ. Namely,

$$\delta_\Lambda h_{ab} := \lambda_o \frac{d}{dt} h_{ab} + \mathcal{L}_\lambda h_{ab}, \tag{46}$$

$$\delta_\Lambda P^{ab} := \lambda_o \frac{d}{dt} P^{ab} + \mathcal{L}_\lambda P^{ab}, \tag{47}$$

$$\delta_\Lambda t := \lambda_o, \quad \text{and} \tag{48}$$

$$\delta_\Lambda x^A := \lambda^a d_a x^A \,, \tag{49}$$

where the time derivative of h_{ab} and P^{ab} is given in (24) and (25) respectively. The evolution of the remaining two fields, P_t and P_A is currently undetermined.

We would now like to determine whether there is a Hamiltonian K_Λ which generates the on-shell evolution δ_Λ given in equations (45-49). Thus we will try to manipulate the symplectic structure into the form

$$\Omega(\delta_\Lambda, \delta) = \delta K_\Lambda + \text{constraints}.$$

In doing this we will be aided by the fact that $\delta_\Lambda P_t$ and $\delta_\Lambda P_A$ are as yet unspecified.

¶ This is really the pre-symplectic structure since the existence of the Hamiltonian and diffeomorphism constraints means that we have not properly isolated the true degrees of freedom. The proper symplectic structure would restrict itself to these real degrees of freedom. That said, we will follow the standard practice and work with the pre-symplectic structure.

The first important step in this process is to evaluate the bulk term in the symplectic structure. Here, we make use of the result of Appendix B:

$$\begin{aligned}
&\int_\Sigma d^3x \left\{(\delta_\Lambda h_{ab})(\delta P^{ab}) - (\delta_\Lambda P_{ab})(\delta h_{ab})\right\} = \\
&\quad \delta \left(\int_\Sigma d^3x \left\{\lambda_o\, N\, \mathcal{H} + (\lambda^a + \lambda_o V^a)\mathcal{H}_a\right\} + \int_{\mathcal{B}} d^2x \sqrt{\sigma}\left[\lambda_o(N\varepsilon - V^a j_a) - \lambda^a j_a\right]\right) \\
&\quad - \int_\Sigma d^3x \left\{\delta(\lambda_o\, N)\mathcal{H} + \delta(\lambda^a + \lambda_o V^a)\mathcal{H}_a\right\} \\
&\quad - \int_{\mathcal{B}} d^2x \sqrt{\sigma}\left[\varepsilon\delta(\lambda_o N) - j_a \delta(\lambda_o V^a + \lambda^a) - \frac{\lambda_o N}{2} s^{ab}\delta\sigma_{ab}\right].
\end{aligned} \tag{50}$$

As one would expect from its general appearance, the derivation of this result is fairly involved. It is discussed in greater detail in the appendix. Here however, we simply note that

$$\begin{aligned}
\varepsilon \;&:= k/(8\pi G), \\
j_a \;&:= \, -2\sigma_{ac}P^{cd}n_d/\sqrt{h}, \text{ and} \\
s^{ab} &:= (1/8\pi G)\left(k^{ab} - (k - n^c a_c)\sigma^{ab}\right),
\end{aligned} \tag{51}$$

where $k = -\sigma^{ab}D_a n_b$ is the extrinsic curvature of $\mathcal{B} \in \Sigma$ (note the sign convention and recall that D_a is the covariant derivative in Σ that is compatible with h_{ab}), $a_c = \frac{1}{N}D_c N$ and G is Newton's constant. The meanings of some of these quantities are easier to see if one thinks of them as being defined in four-dimensional space-time — this perspective is discussed in section 4 following equation (84).

In the last line of (50), there are terms involving the variation of the boundary 2-metric, lapse and shift. However, due to our boundary conditions, we know that δ must act on the lapse, shift, and two-metric like an infinitesimal diffeomorphism in B generated by the vector field

$$\delta X^a = \delta t\, T^a + (\delta x)^a .$$

Then, with the help of Appendix C, we see that because the equations of motion hold at the boundary,

$$\begin{aligned}
&\int_{\mathcal{B}_t} d^2x\, \sqrt{\sigma}\left(\varepsilon \mathcal{L}_{\,\delta X} N - j_a \mathcal{L}_{\,\delta X} V^a - \frac{N}{2} s^{ab}\mathcal{L}_{\,\delta X}\sigma_{ab}\right) = \\
&\qquad \int_{\mathcal{B}_t} d^2x\, (\delta t)\, \frac{d}{dt}(\sqrt{\sigma}[N\varepsilon - V^a j_a]) - \int_{\mathcal{B}_t} d^2x\, (\delta x)^a\, \frac{d}{dt}(\sqrt{\sigma} j_a)\,,
\end{aligned} \tag{52}$$

where the four-dimensional, in-boundary Lie derivatives $\mathcal{L}_T$ from Appendix C are replaced with $\frac{d}{dt}$ from our three-dimensional perspective. We can now use (50) and (52) to rewrite the bulk part of the symplectic structure in (44) to obtain

$$\Omega(\delta_\Lambda, \delta) = \delta\left(\int_\Sigma d^3x\left[(\lambda_o N)\mathcal{H} + (\lambda_o V^a + \lambda^a)\mathcal{H}_a\right]\right)$$

$$
\begin{aligned}
&+\delta\left(\int_{\mathcal{B}} d^2x\,\sqrt{\sigma}\,[\lambda_o(N\varepsilon - V^a\,j_a) - \lambda^a\,j_a] + \lambda_o P_t + \int_{\mathcal{B}} d^2x\,\lambda^A P_A\right)\\
&-\int_\Sigma d^3x\,\left[\delta(\lambda_o N)\mathcal{H} + \delta(\lambda_o V^a + \lambda^a)\mathcal{H}_a\right]\\
&-(\delta\lambda_o)\left[P_t + \int_{\mathcal{B}} d^2x\sqrt{\sigma}\,(N\varepsilon - V^a j_a)\right]\\
&-(\delta t)\left[\delta_\Lambda P_t + \int_{\mathcal{B}} d^2x\,\lambda_o\frac{d}{dt}\left(\sqrt{\sigma}(N\varepsilon - V^a j_a)\right)\right]\\
&-\int_{\mathcal{B}} d^2x\left[\,(\delta\lambda^A)P_A - (\delta\lambda^a)\sqrt{\sigma}j_a\right]\\
&-\int_{\mathcal{B}} d^2x\,(\delta x)^A\left[\delta_\Lambda P_A - \lambda_o\,\frac{d}{dt}\left(\sqrt{\sigma}j_A\right)\right]\,.
\end{aligned}
\tag{53}
$$

This is nearly the desired form. We see that the first two lines are an exact variation, as desired. The third line will vanish provided the usual bulk constraints of general relativity are satisfied. The fourth line is then a new constraint which appears at the boundary and relates P_t to other fields on the boundary. The fifth line gives the evolution of P_t and guarantees that the constraint will continue to hold (these two correspond to the constraint and evolution equation found for p_t in the case of the harmonic oscillator). We would like to write the final two lines in a similar form — a constraint for P_A and an evolution equation which ensures that this constraint is preserved. To this end, we must evaluate $\delta(\lambda^A)$. Making use of (38), it follows that

$$\delta(\lambda^A) = (\delta\lambda)^a(d_a x^A) - \lambda^a d_a(\delta x^A)\,, \tag{54}$$

where as usual the capital Latin index indicates a component of a field in the coordinate system imposed on $\mathcal{B}$, while the lower case index is an abstract tensor index. Therefore, we obtain:

$$\delta(\lambda^A)P_A - (\delta\lambda)^a\sqrt{\sigma}j_a = (P_A - \sqrt{\sigma}j_A)\delta(\lambda^A) - \sqrt{\sigma}j_A\lambda^a d_a(\delta x^A)\,, \tag{55}$$

where $j_A = j_a\left[\frac{\partial}{\partial x^A}\right]^a$. Substituting this expression into (53), doing some integration by parts, and using Stokes theorem, we can rewrite the last two lines as

$$-\int_{\mathcal{B}} d^2x(\delta\lambda^A)\left[P_A - \sqrt{\sigma}j_A\right] - \int_{\mathcal{B}} d^2x\,(\delta x)^A\left[\delta_\Lambda P_A - \delta_\Lambda(\sqrt{\sigma}j_A)\right]\,, \tag{56}$$

where as usual $\delta_\Lambda = \lambda_o\frac{d}{dt} + \mathcal{L}_\lambda$.

In order to simplify matters, let us introduce some notation:

$$H_\Lambda = \int_\Sigma d^3x\,\{\lambda_o(N\mathcal{H} + V^a\mathcal{H}_a) + \lambda^a\mathcal{H}_a\} \tag{57}$$

$$K_t\, = P_t + \int_{\mathcal{B}} d^2x\sqrt{\sigma}[N\varepsilon - V^a j_a] \text{ and} \tag{58}$$

$$L_A = P_A - \sqrt{\sigma}j_A\,. \tag{59}$$

Making use of the new notation, as well as (56), we can rewrite the symplectic structure (53) as

$$\begin{aligned}\Omega(\delta_\Lambda, \delta) = \delta &\left(H_\Lambda + \lambda_o K_t + \int_{\mathcal{B}} d^2x\, \lambda^A L_A \right) \\ &- \int_\Sigma d^3x\, [\delta(\lambda_o N)\mathcal{H} + \delta(\lambda_o V^a + \lambda^a)\mathcal{H}_a] \\ &- (\delta\lambda_o)K_t - \int_{\mathcal{B}} d^2x (\delta\lambda^A) L_A \\ &- (\delta t)(\delta_\Lambda K_t) - \int_{\mathcal{B}} d^2x (\delta x)^A (\delta_\Lambda L_A)\,. \end{aligned} \tag{60}$$

In order for a Hamiltonian generating evolution along Λ to exist, the right hand side of (60) must be an exact variation. Thus, all terms after the first — which is already an exact variation — must vanish. This can be accomplished if the last three lines vanish.Therefore, the evolution along Λ will be generated by a Hamiltonian provided that we are "on-shell," by which we mean:

$$\begin{aligned} \mathcal{H} &= 0 \quad \text{and} \quad \mathcal{H}_a = 0\,; \\ K_t &= 0 \quad \text{and} \quad L_A = 0\,. \end{aligned} \tag{61}$$

The first two expressions are the usual constraints of general relativity, while the last two restrict our evolution to a constraint surface in the extended phase space. They can equivalently be thought of as fixing the values of P_t and P_A to be

$$P_t = -\int_{\mathcal{B}} d^2x \sqrt{\sigma}[N\varepsilon - V^a j_a] \quad \text{and} \quad P_A = \sqrt{\sigma} j_A\,. \tag{62}$$

Finally, the last two terms in (60) define the action of δ_Λ on P_t and P_A. Essentially, it fixes the evolution so that it evolves points on the constraint surface defined by equations (62) into other points on that surface.

Therefore, we have shown that "on-shell," i.e. when (61) is satisfied, the evolution along the vector field Λ is generated by the Hamiltonian K_Λ which is given as:

$$K_\Lambda = \int_\Sigma d^3x\, \{(\lambda_o N)\mathcal{H} + (\lambda^a + \lambda_o V^a)\mathcal{H}_a\} + \lambda_o K_t + \int_{\mathcal{B}} d^2x\, \lambda^A L_A\,. \tag{63}$$

Although we have introduced a coordinate system on the boundary in order to characterize the diffeomorphism ω, the final form of the Hamiltonian is independent of this choice. Specifically, the term $\lambda^A L_A$ will be the same when evaluated in any set of coordinates. As in the previous example, this Hamiltonian is not determined uniquely. However, we shall postpone discussion of the ambiguities to the next subsection.

For the spherically symmetric case, a similar analysis to this one may be found in [7]. In that case, the symmetry means that there is no need to include "angular" coordinates as configuration variables.

3.4. Energy and Angular Momentum

The on-shell value of the Hamiltonian (63) generating evolution along Λ will be zero. To find the energy and angular momentum we must again consider the value of the conjugate momenta rather than the Hamiltonian. Since the equations of motion guarantee that K_t vanishes on-shell, we can use (58) to find the value of P_t. The energy associated to time translation is then simply the negative of this:

$$E_{d/dt} := -P_t = \int_{\mathcal{B}} d^2x \sqrt{\sigma} \left(N\varepsilon + V^a j_a \right) , \tag{64}$$

Similarly, the equations of motion show that L_A vanishes on shell, so that

$$P_A = \sqrt{\sigma} j_A, \tag{65}$$

Combining these two results, we arrive at the expression

$$E_\Lambda = \int_{\mathcal{B}} d^2x \, \left\{ \lambda_o (N\varepsilon + V^a j_a) + \lambda^a j_a \right\} , \tag{66}$$

for the "charge" associated with evolution according to Λ. Specializing to the case of pure time evolution, we obtain (64) which is the energy associated with the time translation $\frac{d}{dt}$. Similarly, given a vector field ϕ tangent to the 3-surface Σ and tangent to $\mathcal{B}$ at the boundary, the angular momentum is given by

$$J_\phi = \int_{\mathcal{B}} d^2x \sqrt{\sigma} \phi^a j_a \, . \tag{67}$$

As in the case of the Hamiltonian, our expressions for the energy and angular momentum associated to the boundary were obtained by making use of a specific coordinatization of the boundary. However, the final results are independent of the choice of coordinates, as one might have hoped.

Finally, we turn to the ambiguity in the Hamiltonian K_Λ and the corresponding freedom in the definition of the energy and angular momentum associated to the boundary. We follow the same procedure that we used in section 2.2 to explore the allowed freedom in the Hamiltonian for the simple harmonic oscillator. Namely, we start by considering adding any functional of the form

$$F = \int_{\mathcal{B}} d^2x f(h_{ab}, P^{ab}, t, P_t, x^A, P_A, \lambda_o, \lambda^A) \, , \tag{68}$$

to K_Λ. However, on requiring that:

(i) the new Hamiltonian still generates the same evolution equations (46-49),

(ii) the new versions of the constraints $K_t = 0$ and $L_A = 0$ remain independent of the Lagrange multipliers λ_o and λ^A (equivalently either energy and angular momentum are independent of these parameters or λ_o and λ^A may be freely chosen without reference to the equations of motion or constraints), and

(iii) the new versions of these constraints are preserved by the new evolution equations for P_t and P_A,

the freedom is greatly reduced. It turns out that subject to these conditions, the only valid Hamiltonian functionals will take the form,

$$K'_\Lambda = K_\Lambda + \lambda_o F(t) + C\,, \tag{69}$$

where $F(t)$ is a free function and C is a free constant.

Furthermore, if we again make the argument that the only physically relevant functions $F(t)$ are those which are associated with the boundary data, then these free functions must take the form

$$F(t) = \int_{\mathcal{B}} d^2x \sqrt{\sigma} f(\sigma_{ab}, N, V^a), \tag{70}$$

where f is a free function of the given fields. Therefore, we find that the energy associated to an evolution along Λ is

$$E'_\Lambda = \int_{\mathcal{B}} d^2x\ [\lambda_o(N\varepsilon + V^a j_a) + \lambda^a j_a] + \int_{\mathcal{B}} d^2x\, \lambda_o \sqrt{\sigma} f(\sigma_{ab}, N, V^a)\,. \tag{71}$$

Readers familiar with [6], will immediately recognize $E_{d/dt}$ with this $F(t)$ as the Brown-York energy, including the usual reference term. In contrast, there is no freedom to add a reference term to the angular momentum expression. Its most general form is

$$J'_\phi = \int_{\mathcal{B}} d^2x \sqrt{\sigma} \phi^a j_a\,, \tag{72}$$

unchanged from equation (67). In the Brown-York derivation a freedom remains in this definition.

Although specific choices of $F(t)$ will not be our major concern here, a few words are in order. The $F(t)$ is a term in the expression for energy that cannot be determined by the formalism. Indeed as far as our formalism is concerned all $F(t)$ are equally valid. Thus, a particular $F(t)$ can be chosen to suit the needs of a particular problem. Popular choices in the literature are usually based on the requirement that $E_{d/dt}$ vanish for some particular space-time (for example flat space or AdS). The reader is directed to any of the papers listed in [8] for further discussion of this point and other applications of this energy expression.

4. Comparison with action arguments

Let us now compare our calculations with those used in the Brown-York derivation of the quasi-local energy. Their original paper [6] pursued two lines of argument, both of which follow from an analysis of the gravitational action I and the boundary conditions which must be imposed on quasi-local boundaries so that the action principal will be well defined. From the action they proceeded in two directions to obtain the quasi-local energy. The first used

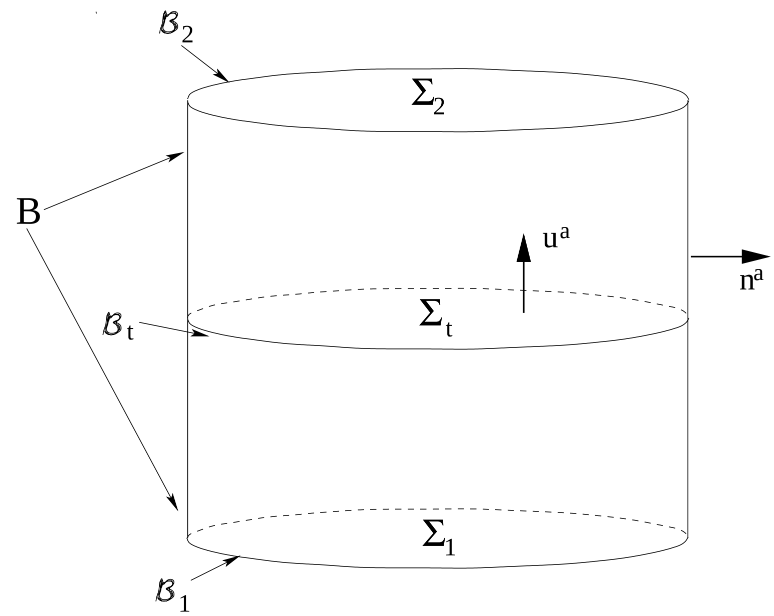

Figure 2. The space-time region M with its boundaries, foliation, and associated vector fields.

Hamilton-Jacobi type arguments and followed a more qualitative path, while the second used a Legendre transform to derive a Hamiltonian functional from the action. In the next few paragraphs we will review the Legendre transform route.

To begin, let us recast the three-dimensional constructions of the previous section into four-dimensional space-time. We consider a region of space-time M that is bounded by a time-like boundary B and two space-like boundaries Σ_1 and Σ_2 as shown in figure 2. The metric over M will be g_{ab} and the associated covariant derivative is ∇_a. The Σ of the last section is now replaced with space-like surfaces Σ_t that foliate M. We assume that Σ_1 and Σ_2 are leaves of that foliation. The Σ_t induce a foliation $\mathcal{B}_t$ of B, and these $\mathcal{B}_t$ replace $\mathcal{B}$. The boundary metrics h_{ab}, γ_{ab}, and σ_{ab} are all induced by g_{ab} and may be calculated as

$$h_{ab} = g_{ab} + u_a u_b, \tag{73}$$

$$\gamma_{ab} = g_{ab} - n_a n_b\,, \quad \text{and} \tag{74}$$

$$\sigma_{ab} = g_{ab} + u_a u_b - n_a n_b, \tag{75}$$

where n_a is the outward-pointing space-like unit normal to B and u_a is the future pointing time-like unit normal to the Σ_t. In defining these metrics we have built in the orthogonality assumption that $u^a n_a = 0$. Allowing non-orthogonal intersections increases the complexity of the calculations but doesn't substantially change the following results [12].

To understand the Brown-York arguments, let us begin with an analysis of the well-

known trace-K action for the quasi-local region M:

$$I = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\mathcal{R} + \frac{1}{\kappa}\int_\Sigma d^3x\sqrt{h}K - \frac{1}{\kappa}\int_B d^3x\sqrt{-\gamma}\Theta\,. \tag{76}$$

In the above, $\mathcal{R}$ is the Ricci scalar corresponding to g_{ab}. Following the Brown-York sign convention for extrinsic curvatures, $K = -\nabla_a u^a$ is the trace of the extrinsic curvature of $\Sigma_{1,2}$ in M while $\Theta = -\nabla_a n^a$ is the trace of the extrinsic curvature of B in M. Further $\int_\Sigma = \int_{\Sigma_2} - \int_{\Sigma_1}$ and $\kappa = 8\pi G$.

The first variation of this action with respect to the metric g_{ab} is (see any standard text on relativity, for example [3]),

$$\delta I = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(G_{ab}+\Lambda g_{ab})\delta g^{ab} + \int_\Sigma d^3x\left(P^{ab}\delta h_{ab}\right) + \int_B d^3x\left(\pi^{ab}\delta\gamma_{ab}\right). \tag{77}$$

Note that in four-dimensions, P^{ab} is not an independent field as in the Hamiltonian formulation, but is instead

$$P^{ab} := \frac{\sqrt{h}}{2\kappa}\left(Kh^{ab} - K^{ab}\right), \tag{78}$$

where $K_{ab} = -h_a{}^c h_b{}^d\nabla_c u_d$ is the extrinsic curvature of $\Sigma_{1,2}$ and K is the trace of K_{ab} as discussed above. Similarly,

$$\pi^{ab} := -\frac{\sqrt{-\gamma}}{2\kappa}\left(\Theta\gamma^{ab} - \Theta^{ab}\right) \tag{79}$$

is an equivalent tensor density defined by the surface B, with $\Theta_{ab} = -\gamma_a{}^c\gamma_b{}^d\nabla_c n_d$ being the extrinsic curvature of B.

Then, if the boundary metrics h_{ab} and γ_{ab} are fixed so that

$$\delta\gamma_{ab} = 0 \quad \text{and} \quad \delta h_{ab} = 0, \tag{80}$$

the variation of the action vanishes on-shell. Note however, that this means that rotations or translations of the boundary are not allowed variations of the action, unless those rotations/translations are generated by Killing vector fields. We will come back to this point momentarily, but for now we note that requiring that the variation of the action vanish and boundary conditions be met gives the Einstein equations in the standard way. Further note that with these boundary conditions, free functionals of the boundary metrics may be added to the action without affecting its first variation — since those terms are fixed, the variations of the functionals will vanish. As such, the exact form of the action is ambiguous and

$$I' = I + I_o[\gamma_{ab}, h_{ab}], \tag{81}$$

is an equally valid action for any functional I_o of the boundary metrics, in the sense that fixing the same boundary conditions and demanding that $\delta I' = 0$ will also generate the usual equations of motion.

Next, in preparation for applying a Legendre transform and so obtaining a Hamiltonian, we take note of the foliation of M. This gives an (imposed) notion of "instants of simultaneity." A time flow is put on M in the guise of a vector field T^a that satisfies $T^a \partial_a t = 1$ and which lies in TB everywhere on B — that is, the time flow generates the boundary. From our orthogonality assumption that $u^a n_a = 0$, we know that u^a is both the unit normal to the Σ_t as well as to the $\mathcal{B}_t$ in B. Thus, we may write

$$T^a = N u^a + V^a \,, \tag{82}$$

for some lapse N and shift vector field V^a that is everywhere an element of $T\mathcal{B}_t$ over B. Making use of this time evolution, we can perform a Legendre transform on the action to obtain:

$$I = \int dt \left\{ -H_t + \int_{\Sigma_t} d^3x \left(P^{ab} \mathcal{L}_T h_{ab} \right) \right\} \tag{83}$$

where

$$H_t = \int_{\Sigma_t} d^3x [N\mathcal{H} + V^a \mathcal{H}_a] + \int_{\mathcal{B}_t} d^2x \sqrt{\sigma}(N\varepsilon - V^a j_a). \tag{84}$$

Now, ε retains its earlier value (see equation (51)) but in the context of this four-dimensional approach j_a can be written as

$$j_a = \frac{1}{8\pi G} \sigma_a{}^b u^c \nabla_b n_c, \tag{85}$$

and so is proportional to the connection on the normal bundle to $\mathcal{B}_t$. Referring back to (51) it is probably also worthwhile to note that in four-dimensions $a_a = u^b \nabla_b u_a$ is the acceleration of the unit normal vector field u_a along its length.

The quasi-local gravitational Hamiltonian for the region of space-time M is given by H_t. However, there is no reason that the functional $I_o[\gamma_{ab}, h_{ab}]$ appearing in (81) should decompose into an integral with respect to t and so give us a Hamiltonian reference term that is local in time — extra assumptions have to be made in order for this to be true. The usual requirement is that I_o be a linear functional of the lapse and shift [6] which guarantees that the energy density ε and angular momentum density j_a are independent of lapse and shift. Then functions of the form

$$H_o = \int_{\mathcal{B}} d^2x \left\{ N f(\sigma_{ab}) + V^a f_a(\sigma_{ab}) \right\} \,, \tag{86}$$

where f is a free function of σ_{ab} and f_a is a free vector valued function of the same, may be added to the Hamiltonian.

Comparison of (64) and (84) shows that the phase space methods of section 3 and the action arguments given here derive the same quasi-local energy, up to assumptions made about the boundary conditions. Recall that in the phase space calculation, we found free

functionals of time only — demanding a connection with the boundary data allowed us to write them in the form of equation (70)

$$F(t) = \int_{\mathcal{B}} d^2x f[\sigma_{ab}, N, V^a]. \tag{87}$$

By contrast, the Brown-York freedom is more general and allows free functionals of the boundary metrics in the action without any need to integrate. It is only after extra assumptions that they can be broken up into Hamiltonian reference terms that are independent of local details of the geometry of B. This difference in boundary terms arises because the boundary conditions that we have imposed are not the same. The Brown-York conditions are more restrictive than ours (and so allow more freedom in the free functionals).

To obtain a fairer comparison, let us weaken the boundary conditions in the action formulation so that they are equivalent to those used in section 3.

(i) Instead of rigidly fixing the metrics on the boundaries, we allow variations which are foliation preserving translations/rotations of the boundary. Thus, the overall geometry of the boundaries will be fixed, although particular features of the geometry will not be fixed to particular points of the manifold. To do this, we allow the variations to act as diffeomorphisms on the boundary metrics, with the restriction that they should map B into itself. Then,

$$\delta\gamma_{ab} = \mathcal{L}_Y \gamma_{ab} \quad \text{and} \quad \delta h_{ab} = \mathcal{L}_Z h_{ab}, \tag{88}$$

for some vectors $Y^a \in TB$ and $Z^a \in T\Sigma$ for which $Y^a = Z^a \in \mathcal{B}_{1,2}$ on those corner two-surfaces.

(ii) At the same time, we impose the new condition that the diffeomorphism constraint should hold on the boundary surfaces Σ_1, Σ_2, and B. That is

$$D_a P^{ab} = 0 \quad \text{and } \triangle_a \pi^{ab} = 0, \tag{89}$$

where, as before, D_a and $\triangle_a$ are the induced covariant derivatives on $\Sigma_{1,2}$ and B respectively. Of course, these constraints hold automatically on-shell. In the action formulation however, variations are not restricted to being on-shell and we will need to impose these conditions in the following calculation.

Then, with these new conditions:

$$\begin{aligned} \delta I &= \int_B d^3x\, \pi^{ab} \mathcal{L}_Y \gamma_{ab} + \int_{\Sigma_2 - \Sigma_1} d^3x\, P^{ab} \mathcal{L}_Z h_{ab} \\ &= 2\int_B d^3x\, \pi^{ab} \triangle_a Y_b + \int_{\Sigma_2-\Sigma_1} d^3x\, P^{ab} D_a Z_b \\ &= 2\int_B d^3x\, \triangle_a(\pi^{ab} Y_b) + \int_{\Sigma_2-\Sigma_1} d^3x\, D_a(P^{ab} Z_b) \\ &= 2\int_{\mathcal{B}_2-\mathcal{B}_1} d^2x \sqrt{\sigma} \left[-\frac{1}{\sqrt{-\gamma}} u_a \pi^{ab} Y_b + \frac{1}{\sqrt{h}} n_a P^{ab} Z_b \right] \end{aligned} \tag{90}$$

$$= \frac{1}{\kappa}\left(\int_{\mathcal{B}_2-\mathcal{B}_1} d^2x\sqrt{\sigma}Y^b\left[u^a\nabla_b n_a + n^a\nabla_b u_a\right]\right)$$
$$= 0.$$

The second line applies the representation of a Lie derivative in terms of the covariant derivatives $\triangle_a$ and D_a of B and $\Sigma_{1,2}$ respectively. The third uses the fact that the diffeomorphism constraint holds on the boundary. The fourth integrates bulk terms out to the boundaries of B and $\Sigma_{1,2}$, while the fifth uses the fact that $Y^a = Z^a \in T\mathcal{B}_{1,2}$ on those boundaries. Thus, with these modified boundary conditions, the action also vanishes on-shell.[+]

Given these boundary conditions, there are other actions for which $\delta I = 0$ and also give the same equations of motion. In particular, consider any reference term which is of the form

$$I_o = -\int_{t_1}^{t_2} dt F(t), \tag{91}$$

where t is some labeling of the foliation which is equal to t_1 on Σ_1 and t_2 on Σ_2. $F(t)$ is a functional that depends on the foliation surface only. Then $\delta I_o = 0$ for the variations that we have considered. On applying the Legendre transform, we will find that for $I + I_o$ the Hamiltonian becomes

$$H_t = F(t) + \int_{\Sigma_t} d^3x\left[N\mathcal{H} + V^a\mathcal{H}_a\right] + \int_{\mathcal{B}_t} d^2x\sqrt{\sigma}(N\varepsilon - V^a j_a)\,, \tag{92}$$

which is the same form that we found with our phase space analysis. Thus, when we impose the same boundary conditions for the two approaches, we obtain the same value for the Hamiltonian functional — as would be expected. As we have seen above, with the standard Brown-York conditions which fix the metric at each point in the boundary the free functionals can depend on specific local features of the intrinsic geometry (equivalently, we could think of a coordinate system imposed on the boundary with the functional being allowed to depend on the coordinate system as well as the metric). With the new boundary conditions however, that freedom has changed and the free functionals can only depend on global geometric features (for example integrals of geometric invariants over the boundary).

5. Discussion

In this paper we have introduced a phase space description of general relativity in a manifold with boundary. The boundary is equipped with a preferred foliation and a metric which is fixed up to diffeomorphisms which preserve this foliation. Therefore, we must allow

[+] The essentials of this proof that the action is invariant with respect to diffeomorphisms that map the boundary into itself may be found in [14], where they occur in the course of a demonstration that the Bianchi (and other) identities may be derived from the action principal.

phase space variations which act as diffeomorphisms of the boundary metric. This can be accomplished by introducing a coordinatization of the boundary and extending the phase space to include both the time and spatial coordinates of the boundary and their conjugate momenta. Then, in the extended phase space, we are able to discuss variations which induce diffeomorphisms of the boundary, even if they are not symmetries of the boundary metric. In particular, time evolution is an allowed variation even if the boundary metric is time dependent. Similarly, rotation along a vector field which is *not* a Killing vector of the boundary metric is also permitted.

Furthermore, we are able to associate "conserved charges" with these motions. In the case where the motion is not a symmetry of the boundary, the "conserved charge" will not be constant. Hence, the conserved charges cannot be equal to the on-shell value of the Hamiltonian (which is by definition a constant). Instead, the on-shell values of the momenta conjugate to the coordinates give the "conserved charges" associated with these motions. In particular, if the variation induces a time translation at the boundary, the corresponding conjugate momentum is equal to the energy of the boundary. Similarly if the variation produces a rotation of the boundary, the conjugate momentum is the angular momentum of the boundary. These quantities need not be conserved: they can vary as you move up the boundary. In addition, the final expressions for the energy and angular momentum are *independent* of the choice of coordinate system of the boundary, as one would hope.

The expressions for energy and angular momentum obtained from the Hamiltonian formulation are the same as those obtained in the Brown–York formalism (up to the free functionals where a difference arises due to the slightly different boundary conditions). This is unsurprising since both methods describe the same situation, just in a different language. In particular, the energy obtained is not uniquely fixed and there is the freedom to add to it a function of the boundary data. When we harmonize boundary conditions as we did in the last section, this is the same freedom as found in the Brown–York formalism and can be fixed in the usual manner. It should be possible to extend the formalism presented here to the case of non-orthogonal boundaries as considered in [12], and work is already under way to do so. * Additionally, we have required throughout that variations preserve the foliation of space-time. It is likely that this condition can also be weakened or removed entirely.

The approach we have described above should have several applications. Indeed, it was initially motivated by a desire to extend the isolated horizon framework [15] to include the physically interesting case of dynamical horizons. In order to describe these horizons within a Hamiltonian framework, it is necessary to allow for a varying black hole mass and angular momentum. Making use of the extended phase space methods introduced here will make this possible [16]. A second application would be to consider the Hamiltonian formulation of

* In fact, in his alternative formulation, Kijowski [5] has already shown that the non-orthogonal "angle" parameter may be included as a configuration variable in the phase space.

general relativity with a boundary at null infinity. Again, the mass and angular momentum are not constant as they can be radiated away. Therefore, the framework here would again be relevant. It would be interesting to recast the results of Wald and Zoupas [17] in terms of the phase space description presented here.

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Appendix A. Constraints and time evolutions

For completeness we include the expressions for $[h_{ab}]_{(N,V)}$, $[P^{ab}]_{(N,V)}$, $\mathcal{H}$, and $\mathcal{H}_b$ (in the absence of matter and a cosmological constant). Keep in mind that the tensor (and tensor density) fields in these expressions all live on Σ and its assorted tensor bundles. Further, D_a is the intrinsic covariant derivative over Σ and $\kappa = 8\pi G$, where G is the gravitational constant.

$$[h_{ab}]_{(N,V)} \equiv \frac{4\kappa N}{\sqrt{h}}[P_{ab} - \frac{1}{2}P h_{ab}] + \mathcal{L}_V h_{ab}, \tag{A.1}$$

$$\begin{aligned}\left[P^{ab}\right]_{(N,V)} \equiv & -\frac{\sqrt{h}}{2\kappa}\left(N\, {}^{(3)}G^{ab} - \left[D^a D^b N - h^{ab} D_c D^c N\right]\right) + \mathcal{L}_V P^{ab} \\ & + \frac{N\kappa}{\sqrt{h}}\left([P^{cd}P_{cd} - \frac{1}{2}P^2]h^{ab} - 4[P^{c(a}P_c{}^{b)} - \frac{1}{2}PP^{ab}]\right),\end{aligned} \tag{A.2}$$

$$\mathcal{H} \equiv -\frac{\sqrt{h}}{2\kappa}R + \frac{2\kappa}{\sqrt{h}}\left(P^{ab}P_{ab} - \frac{1}{2}P^2\right), \quad \text{and} \tag{A.3}$$

$$\mathcal{H}_b \equiv -2D_b P_a{}^b. \tag{A.4}$$

In the above ${}^{(3)}G_{ab} = R_{ab} - \frac{1}{2}R g_{ab}$, where R_{ab} is the Ricci tensor on Σ and R is its contraction with the metric h_{ab}.

Full derivations of the above results from the four-dimensional geometry of solutions to the Einstein equations can be found in most standard relativity texts, or using this notation

in [13]. Essentially though, they come from thinking of Σ as one leaf in the foliation of a solution of the Einstein equations (M, g_{ab}) into space-like hypersurfaces (as discussed in section 4). If u^a is the future pointing unit normal to Σ, the time evolution given by (N, V^a) corresponds to that generated by the vector field $T^a = Nu^a + V^a$. Further P^{ab} is no longer an independent variable, but instead is closely connected to the extrinsic curvature of Σ in M (see equation (78)).

The Einstein equations tell us that in empty space, $G_{ab} = 0$. Then, the Hamiltonian constraint (A.3) is equivalent to the statement $G_{ab}u^a u^b = 0$, the diffeomorphism constraint (A.4) is equivalent to $h_a^b G_{bc} u^c = 0$, and the evolution equation for P^{ab} (A.2) is equal to $\mathcal{L}_T P^{ab}$ with $h_a^c h_b^d G_{cd} = 0$ being used to rewrite the result in terms of quantities defined entirely on Σ. Finally, the evolution equation for h_{ab} is equivalent to $\mathcal{L}_T h_{ab}$.

Appendix B. Variational calculation

In this appendix we show that if λ_o is any function on Σ which is constant on $\mathcal{B}$ and $\lambda^a \in T\Sigma$ is any vector field that lies in $T\mathcal{B}$ on $\mathcal{B}$ so that

$$\delta_\Lambda P^{ab} \equiv \lambda_o [P^{ab}]_{(N,V)} + \mathcal{L}_\lambda P^{ab} \quad \text{and} \quad \delta_\Lambda h_{ab} \equiv \lambda_o [h_{ab}]_{(N,V)} + \mathcal{L}_\lambda h_{ab} \, , \tag{B.1}$$

then

$$\begin{aligned} &\int_\Sigma d^3x \left[(\delta_\Lambda h_{ab})(\delta P^{ab}) - (\delta_\Lambda P_{ab})(\delta h_{ab}) \right] = \\ &\quad \delta \left(\int_\Sigma d^3x \left[\lambda_o N \mathcal{H} + (\lambda^a + \lambda_o V^a)\mathcal{H}_a \right] + \int_\mathcal{B} d^2x \sqrt{\sigma} \left[\lambda_o (N\varepsilon - V^a j_a) - \lambda^a j_a \right] \right) \\ &\quad - \int_\Sigma d^3x \left[\delta(\lambda_o N)\mathcal{H} + \delta(\lambda^a + \lambda_o V^a)\mathcal{H}_a \right] \\ &\quad - \int_\mathcal{B} d^2x \sqrt{\sigma} \left[\varepsilon \, \delta(\lambda_o N) - j_a \, \delta(\lambda_o V^a + \lambda^a) - \frac{\lambda_o N}{2} s^{ab} \delta\sigma_{ab} \right] , \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} \varepsilon \; &:= k/(8\pi G), \\ j_a \; &:= -2\sigma_{ac} P^{cd} n_d / \sqrt{h}, \quad \text{and} \\ s^{ab} &:= (1/8\pi G) \left(k^{ab} - (k - n^c a_c)\sigma^{ab} \right) . \end{aligned}$$

$k_{ab} = -\sigma_a{}^c \sigma_b{}^d D_c n_d$ is the extrinsic curvature of $\mathcal{B}$ in Σ (note the sign convention), $k = \sigma^{ab} k_{ab}$, and $a_c = \frac{1}{N} D_c N$.

The calculations needed to obtain this result are quite lengthy, but luckily the bulk of them may be found in previous papers. The key result may be found in [13] or [14], and says that for a general variation δ (there is no restriction to it being on-shell),

$$\delta \left(\int_\Sigma d^3x [N\mathcal{H} + V^a \mathcal{H}_a] \right) = \tag{B.3}$$

$$\int_\Sigma d^3x \left\{\mathcal{H}\delta N + \mathcal{H}_a \delta V^a + [h_{ab}]_{(N,V)}\delta P^{ab} - [P^{ab}]_{(N,V)}\delta h_{ab}\right\}$$
$$- \int_{\mathcal{B}} d^2x \left\{ N\delta(\sqrt{\sigma}\varepsilon) - V^a\delta(\sqrt{\sigma}j_a) + \frac{1}{2}\sqrt{\sigma}Ns^{ab}\delta\sigma_{ab}\right\},$$

Now, a careful examination of the calculations found in the above sources, shows that if we replace the N and V^a on the left hand side of the equation with more general terms — for example any scalar field α over Σ and any vector field β^a over Σ that is restricted to be parallel to $\mathcal{B}$ on that boundary, then the expression changes to become

$$\begin{aligned}\delta\left(\int_\Sigma d^3x[\alpha\mathcal{H} + \beta^a\mathcal{H}_a]\right) &= \int_\Sigma d^3x\left\{\mathcal{H}\delta\alpha + \mathcal{H}_a\delta\beta^a\right\} \\ &+ \int_\Sigma d^3x \left\{(\delta_\xi h_{ab})(\delta P^{ab}) - (\delta_\xi P^{ab})(\delta h_{ab})\right\} \\ &- \int_{\mathcal{B}} d^2x \left\{\alpha\,\delta(\sqrt{\sigma}\varepsilon) - \beta^a\,\delta(\sqrt{\sigma}j_a) + \frac{1}{2}\sqrt{\sigma}\alpha s^{ab}\delta\sigma_{ab}\right\},\end{aligned} \tag{B.4}$$

where $\xi = (\alpha/N)\frac{d}{dt} + \mathcal{L}_{\beta-(\alpha/N)V}$ and $\delta_\xi P^{ab}$ and $\delta_\xi h_{ab}$ are defined in an analogous way to $\delta_\Lambda P^{ab}$ and $\delta_\Lambda h_{ab}$ in equation (B.1). Therefore, in order to obtain the desired result (B.2), we must take

$$\alpha = \lambda_o N \quad , \quad \beta^a = \lambda^a + \lambda_o V^a ,$$

and re-express the surface term as

$$\int_{\mathcal{B}} d^2x \left\{\alpha\,\delta(\sqrt{\sigma}\varepsilon) - \beta^a\delta(\sqrt{\sigma}j_a) + \frac{1}{2}\sqrt{\sigma}\alpha s^{ab}\delta\sigma_{ab}\right\} =$$
$$\delta\left(\int_{\mathcal{B}} d^2x\sqrt{\sigma}\,[\alpha\varepsilon - \beta^a j_a)]\right) - \int_{\mathcal{B}} d^2x\sqrt{\sigma}\left[\varepsilon\delta(\alpha) - j_a\delta(\beta^a) - \frac{\alpha}{2}s^{ab}\delta\sigma_{ab}\right],$$

to obtain (B.2), the desired result.

Appendix C. The diffeomorphism constraint on the boundary

In this appendix we consider a solution to the full four-dimensional Einstein equations over a region M like that discussed at the beginning of section 4. We show that if

(i) the diffeomorphism constraint holds on a time-like surface B that has intrinsic metric γ_{ab},

(ii) B is foliatied with closed, space-like, two-surfaces $\mathcal{B}_t$ and there exists a vector field T^a such that $T^a[dt]_a = 1$, and

(iii) X^a is a vector field taking the form $X^a = X_o T^a + \hat{X}^a$ where X_o is a function of t alone, while $\hat{X}^a$ is any non-singular vector field over B that is everywhere parallel to the $\mathcal{B}_t$ surfaces,

then

$$\int_{\mathcal{B}_t} d^2x \left(\varepsilon \mathcal{L}_X N - j_a \mathcal{L}_X V^a - \frac{N}{2} s^{ab} \mathcal{L}_X \sigma_{ab} \right) = \int_{\mathcal{B}_t} d^2x \, X_o \, \mathcal{L}_T \left(\sqrt{\sigma}[N\varepsilon - V^a j_a] \right) - \int_{\mathcal{B}_t} d^2x \, \hat{X}^a \, \mathcal{L}_T \left(\sqrt{\sigma} j_a \right) . \tag{C.1}$$

where N and V^a are the usual lapse and shift defined so that if u^a is the forward-pointing unit normal to the $\mathcal{B}_t$ (that is $T^a u_a < 0$), then $T^a = N u^a + V^a$. σ_{ab} is the induced two-metric on the $\mathcal{B}_t$ surfaces, and ε, j_a and s_{ab} are defined in (51).

We begin by introducing the conjugate variable to γ_{ab}, namely π^{ab} which is given by:

$$\pi^{ab} = \frac{\sqrt{-\gamma}}{16\pi G} \left(\Theta^{ab} - \Theta \gamma^{ab} \right) , \tag{C.2}$$

where $\Theta_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_c n_d$ is the extrinsic curvature of B with respect to the unit normal n^a. Then, we can write ε, j_a and s_{ab} in terms of π^{ab} so that

$$\begin{aligned} \varepsilon &= -2\pi^{ab} u_a u_b / \sqrt{-\gamma} \, , \\ j_a &= 2\sigma_{ab} \pi^{bc} u_c / \sqrt{-\gamma} \, , \text{ and} \\ s_{ab} &= -2\sigma_a{}^c \sigma_b{}^d \pi^{cd} / \sqrt{-\gamma} \, . \end{aligned}$$

Now, taking $\triangle_a$ as the covariant derivative on B we first note that the diffeomorphism constraint implies that

$$\triangle_a \pi^{ab} = 0 \, ,$$

and therefore

$$\int_{\mathcal{B}_t} d^2x \, \triangle_a (\pi^{ab} X_b) = \frac{1}{2} \int_{\mathcal{B}_t} d^2x \, \pi^{ab} \, \mathcal{L}_X \gamma_{ab} \, . \tag{C.3}$$

The result (C.1) arises when we break up each side of the expression (C.3) into parts parallel and perpendicular to the foliation. To that end note that

$$\gamma_{ab} = \sigma_{ab} - u_a u_b \quad \text{and} \quad \sqrt{-\gamma} = N\sqrt{\sigma} \, .$$

Then, working on the left hand side we can show that

$$\sqrt{-\gamma} \triangle_a (\pi^{ab} X_b / \sqrt{-\gamma}) = -\mathcal{L}_T (\sqrt{\sigma} u_a \pi^{ab} X_b / \sqrt{-\gamma}) + \mathcal{L}_V (\sqrt{\sigma} u_a \pi^{ab} X_b / \sqrt{-\gamma}) + \sqrt{\sigma} \, d_a (N \sigma^a{}_b \pi^{bc} X_c / \sqrt{-\gamma}) \, . \tag{C.4}$$

Since the surfaces $\mathcal{B}_t$ are closed, when we integrate this expression only the first term on the right hand side survives, giving

$$\int_{\mathcal{B}_t} d^2x \, \triangle_a (\pi^{ab} X_b) = \frac{1}{2} \int_{\mathcal{B}_t} d^2x \, \mathcal{L}_T \left(\sqrt{\sigma} (X_o \, [N\varepsilon - V^a j_a] - \hat{X}^a j_a) \right) . \tag{C.5}$$

Let us now turn our attention to the right-hand side of equation (C.3). We begin by recalling that $u_a = -N[dt]_a$. Therefore, with X_o constant on each slice and $\mathcal{L}_X (dt_a) = (\mathcal{L}_T X^0) dt_a$ it is simple to show that

$$\mathcal{L}_X u_a = \left(\frac{\mathcal{L}_X N}{N} + \mathcal{L}_T X_o \right) u_a \, . \tag{C.6}$$

In particular, it then follows immediately that $\sigma^{ab}\mathcal{L}_X u_a = 0$. Finally, by decomposing the metric γ_{ab} in terms of u_a and σ_{ab} we obtain:

$$\begin{aligned} \pi^{ab}\mathcal{L}_X\gamma_{ab} &= \sqrt{-\gamma}\left\{\varepsilon\, u^a\,\mathcal{L}_X u_a + j_a(\mathcal{L}_u X^a + \sigma^{ab}\mathcal{L}_X u_b) + \frac{1}{2}s^{ab}\mathcal{L}_X\sigma_{ab}\right\} \\ &= \sqrt{\sigma}\left(\varepsilon\mathcal{L}_X N - j_a\mathcal{L}_X V^a - \frac{N}{2}s^{ab}\mathcal{L}_X\sigma_{ab}\right) - \sqrt{\sigma}j_a\mathcal{L}_T\hat{X}^a \\ &\quad + \sqrt{\sigma}(N\varepsilon - V^a j_a)\mathcal{L}_T X_o\,. \end{aligned} \tag{C.7}$$

Equations (C.3), (C.5), and (C.7) can then be combined to give the promised result:

$$\begin{aligned} &\int_{\mathcal{B}_t} d^2x\left(\varepsilon\mathcal{L}_X N - j_a\mathcal{L}_X V^a - \frac{N}{2}s^{ab}\mathcal{L}_X\sigma_{ab}\right) = \\ &\qquad \int_{\mathcal{B}_t} d^2x\, X_o\,\mathcal{L}_T\left(\sqrt{\sigma}[N\varepsilon - V^a j_a]\right) - \int_{\mathcal{B}_t} d^2x\,\hat{X}^a\,\mathcal{L}_T\left(\sqrt{\sigma}j_a\right)\,. \end{aligned} \tag{C.8}$$

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